Invited Lecture

Teaching Maths in Secondary (Middle and High) Schools: Complex Strategy and its Successful Implementation

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ABSTRACT This article deals with Maths education in the Middle and High School in Kharkiv City (Ukraine) and in Academic Gymnasium No. 45 in particular. It shows the whole structure of education and the ways of motivation for learning Maths at the high level by students. It also shows the obvious success of the strategy of complex Maths teaching and analyzes its positive results for the last 25 years.

Keywords: Maths education; Maths competitions; Maths Olympiads; Development of critical thinking; Students' scientific research.

1. Introduction

This article deals with Maths education in Secondary (Middle and High) Schools in Kharkiv City (Ukraine) and in the specialized school — Academic Gymnasium No. 45 in particular. It shows the whole structure of education in the chain country-city-school-class and ways to motivate students to learn higher level Maths. It also shows the obvious success of the strategy of complex Maths teaching and analyses its positive results for the last 25 years.

There are many countries with famous scientific schools and traditions in Mathematics. However, our world is changing rapidly. It takes teachers a long time to motivate their students to study science and its applications at universities. So, Maths teachers should try to make efforts to encourage and motivate young people to get knowledge, the sooner the better. With the development of our society this is not an easy task, because of a lot of temptations far from science and learning.

The following information concerns the unique experience in Kharkiv city and at our school for a regular creation of student's motivation for deep Maths learning.

2. The Unique Experience in Kharkiv City

2.1. Maths education in Kharkiv city

Kharkiv is one of the largest scientific centers of Ukraine (East Europe). There were 3 Nobel Prize winners in Kharkiv. It is the place where an atomic nucleus was split one of the first times in history. The scientists of Kharkiv University cooperated with

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students of city schools. But only at the beginning of 1980s we managed to create extra-curricular Maths courses in the city. So, a great number of students were involved and they started learning additional Maths topics. It was wide spread in the 1990s and hundreds of students 5–17 years joined these courses. Today there are 3 big extra Maths educational centers with about 5 000 students (there are about 114 000 students in Kharkiv schools).

As a result, a lot of students became the winners in Maths Olympiads. It is also important, that learning Maths is becoming more and more popular, prestigious and trendy. Furthermore, our city council provides talented students with different grants and scholarships and reflects their success in Mass Media. Thus, we have the basis for children learning Maths in Elementary, Middle and High Schools. Unfortunately, there is a lack of experienced Maths teachers in Kharkiv to support students' interest in Maths and to develop their abilities.

2.2. Maths education in Academic Gymnasium No. 45 and general principles

Speaking about Academic Gymnasium No. 45 we have our own unique system of such complex Maths development of students. We have been working on this task for about 25 years. The following information is about the special features of our teaching concept and how to apply this system successfully in other schools. The main idea is that successful Maths teaching at school should be complex [2] and includes the following:

1) Teaching basic knowledge at lessons via heuristic methods,
2) Preparing for Maths competitions at lessons and extra-curricular lessons as well,
3) Preparing science projects with students, under the leadership of Mathematicians in particular,
4) Development of critical thinking skills and a scientific mindset, realization of the importance of Maths and its connection to modern Computer Science.

Maths teachers should not only give basic knowledge but also inspire students to solve problems, including playing and small competitions among themselves. So, I believe it is very important to recognize mathematically gifted children, pay them special attention, involve them into creative Maths discoveries at lessons, additional lessons, tutorials, and organize their attendance in Maths development centers. Due to participation in different competitions, scientific contests and conferences a lot of students consider Maths not only as a strict and boring school subject. It is also important to influence not only advanced students, but the whole class.

Unfortunately, the majority of schools are oriented on only one direction, such as: strict following the course program, preparing for final and entrance exams, or work only with gifted students. But following this strategy we have a tendency of students' losing interest for Maths, and Maths teaching is becoming less effective. Let children

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[2] A. Kryzhanovskiy (2015)

do what they like, but under supervision. Most of them like playing so they can play Maths games. If they like to compete, give them a chance to do it at lessons. If they are fond of gadgets, they can use them for Maths modeling!

It is important for teachers to avoid putting certain boundaries on students Maths development. For example, we have to publish popular Maths books as much as possible oriented not only for the top 5% students, but with a style such that it is available for most of students and their parents. Unfortunately, we have a gap in these types of Maths literature. On the one hand, some of these books are for very low students, and on the other hand there are books for advanced students only. It looks like Maths books are written in absolutely different styles, end even of different subjects. As a result, the majority of students can't find the appropriate books. It's important for children to know that lessons at school, popular science books, and additional Maths literature for Olympiad participants are all aspects of the same science — Maths!

2.3. Selective exams

My work with the students of Academic Gymnasium No. 45 starts at their entrance exams after the 4th grade. These selective exams give us a chance to find out students with good mathematical abilities. But it does not mean that all gifted children can pass these exams, and all of the selected students will connect their life with Maths in the future. Though it is very important to develop the students' personalities and to give them an opportunity for creative research in an appropriate surrounding. There are two steps in our entrance exams:

1) A competition for students of Grades 3−6, which is called "The world of Maths",
2) Entrance tests "Student of Gymnasium".

The first step is oriented on finding out most of the gifted students with a strong and special mindset. As usual, these students have been attending different city Maths courses for a while, but some of them are real prodigies. The second step is based on testing the learned basic knowledge of the elementary school and students' abilities of applying it in unusual situations. The winners of these competitions form a special class. In fact, most of these children are really good at Maths, so my goal as a teacher is to develop their abilities based on certain topics in Maths.

2.4. Three parts of Maths education

2.4.1. Usual lessons and the development of students' critical thinking

The further organizational work is conducted in three directions. The first one is making lessons in which compulsory topics are combined with solving Olympiad problems and tasks for the development of their thinking skills. It takes the same time as usual, because gifted students are very quick in standard methods of solving problems and need a challenge. It is very effective to organize a group work at lessons while solving multi-case problems especially in geometry. It influences the development of students' critical thinking and teamwork skills.

For example, let us consider the following problem[3].

Problem 2.1 In *the given parallelogram there is a height from the vertex of the obtuse angle. This height divides the opposite side by two segments with the ratio 1:7. Find the ratio of the two segments of the diagonal obtained by the intersection with the given height.*

Consider two triangles in Fig. 1: ΔAFE and ΔCFB: $\angle AFE = \angle BFC$ as vertical; $\angle FAE = \angle BCF$ as interior alternate angles for $BC \parallel AD$, AC transversal. So $\Delta AFE \sim \Delta CFB$. Hence, $\frac{AF}{FC} = \frac{AE}{BC} = \frac{1}{1+7} = \frac{1}{8}$.

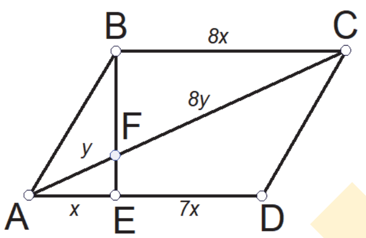

Fig. 1. The first case of the problem 2.1

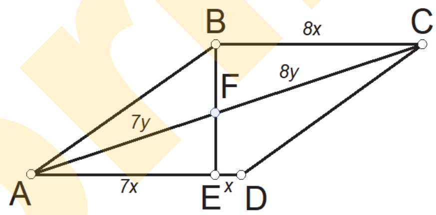

Fig. 2. The second case of the problem 2.1

Let us analyze how our figure corresponds to the given conditions. In the given conditions we have the ratio of two parts of the side of the parallelogram, but nothing about the order of the two parts. So, we have another case in this problem. Let us look at Fig. 2.

In the same way, we have a similarity of 2 triangles *AFE* and *CFB*, and a corresponding proportion: $\frac{AF}{FC} = \frac{AE}{BC} = \frac{7}{1+7} = \frac{7}{8}$.

Hence, in this case we have another answer.

It seems now, that we considered all possible cases. But no! We have two more cases for the location of the given height. Let us consider two new situations on the Fig. 3 and Fig. 4 (on the next page):

These cases are interesting due to the fact, that the height is drawn to one side, but divides proportionally another side.

Let us consider these two situations. In both we have two similar triangles: $\Delta BIC \sim \Delta EID$. Indeed, $\angle BIC = \angle EID$ as vertical, $\angle BCI = \angle EDI$ as interior alternate

[3] O. Kryzhanovskiy (2016)

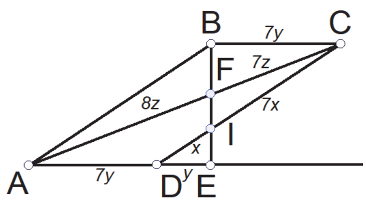

Fig. 3. The third case of the problem 2.1

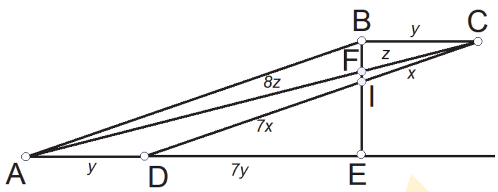

Fig. 4. The fourth case of the problem 2.1

angles for $BC\,//\,AD$, CD transversal. So, $\frac{CI}{DI}=\frac{BC}{DE}$. With the same way we can obtain that $\Delta AFE \sim \Delta CFB$. Hence, $\frac{AF}{FC}=\frac{AE}{BC}$.

These cases are interesting due to the fact, that the height is drawn to one side, but divides proportionally another side.

Let us consider these two situations. In both we have two similar triangles: $\Delta BIC \sim \Delta EID$. Indeed, $\angle BIC = \angle EID$ as vertical, $\angle BCI = \angle EDI$ as interior alternate angles for $BC\,//\,AD$, CD transversal. So, $\frac{CI}{DI}=\frac{BC}{DE}$. Similarly, we have $\Delta AFE \sim \Delta CFB$. Hence, $\frac{AF}{FC}=\frac{AE}{BC}$.

So, in the 3rd case $\frac{AF}{FC}=\frac{8}{7}$, and in the 4th case $\frac{AF}{FC}=\frac{8}{1}$.

As a result, we have 4 cases in this problem, and two answers only — 8:7 and 8:1.

Now Let us look at an example of another interesting side of complex Maths teaching — connections with other subjects. The following shows a commonality between the AM-GM inequality and electric circuits.

Given two electric circuits, where both of them consist of one battery and two resistors (Fig. 5 and Fig. 6).

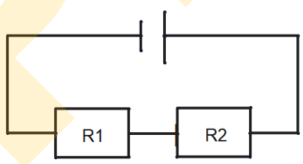

Fig. 5. Two resistors are connected in series with the battery.

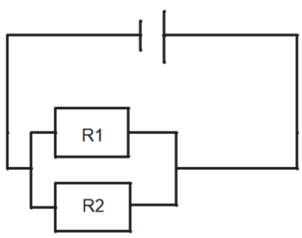

Fig. 6. Two resistors are connected in parallel with the battery.

In the first circuit two resistors are connected in series with the battery. In the second circuit the same resistors are connected in parallel with the battery. Find the minimum ratio between the total resistances in these two circuits.

Due to physical laws for total resistance, we have that in the first circuit $R' = R_1 + R_2$, and in the second one $R'' = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$. But how can we compare these results?

Let us use now the AM-GM inequality. With that we obtain the following:

$$\frac{R'}{R''} = (R_1 + R_2) \cdot \left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{(R_1 + R_2)^2}{R_1 R_2} = 4 \cdot \frac{\left(\frac{R_1 + R_2}{2}\right)^2}{R_1 R_2} \geq 4 \cdot \frac{R_1 R_2}{R_1 R_2} = 4.$$

With the smallest ratio of 4 the resistance for the series circuit is at least 4 times the equivalent of the resistance of the parallel circuit.

2.4.2. Extra maths classes

The second direction is based on working with the children of the special class at our additional Maths classes "Solving of Maths Olympiad problems" systematically. At these lessons we study applying certain mathematical methods on appropriate examples adapted for children. As usual, the students enjoy this kind of activity, which content is related to serious mathematics, but it is unusual and often resembles as a game. These extra lessons exist in all grades from 5 to 11 (12), and are held once a week. The most important thing in this approach is adapting famous methods to develop the mathematical skills of children to school requirements.

2.4.3. Individual and group lessons

The third step is individual and group lessons prepare the most gifted students for Olympiads, competitions and scientific conferences. Very often I'm not just a teacher, but also, I partner up with my students while solving actual problems. As usual, a good competition spirit and students Maths ambitions give a quite positive dynamic in the learning process. It is also important that all of the children are involved in creative work, in spite of changing forms and methods of learning.

It is very difficult to divide the topics which are learned at usual lessons and which are used only at Olympiads. Thus, I try to include Olympiad and research problems at our lessons as much as possible. Preparing for Maths competitions we try to discover how usual methods of solving problems can be applied at Maths Olympiads.

So, it is obvious that teachers, their students and their parents should have close contact to each other. In fact, teachers and students communicate at lessons, preparing for Olympiads, visiting Maths Festivals, at conferences, scientific competitions and at summer Maths schools. Thus, at Academic Gymnasium No. 45 in Kharkiv we are going to have the 15th year of our traditional summer school with the profile "Maths and Computer Science". One of the main goals of this school is to stimulate the students' motivation in both Maths and Computer Science studies, in the form of creative communication between students and their teachers.

2.4.4. Three types of Maths competitions

To involve students in creative Maths learning I try to organize their participation in competitions of 3 types. The first type consists of competitions, available for all students, such as "The Kangaroo", "The Championship of Maths logical solving problems" (organized by France). The next one consists of competitions for mathematically gifted students, ready for intellectual and mental fighting. For instance, there is a system of national Olympiads in Ukraine, which includes 4 steps, with the final national Ukrainian Olympiad. Our students take also part in the "International Mathematical Tournament of Towns", "Maths fest" and Olympiad named after Euler (organized by Russia). The third type is the most difficult and consists of individual and group competitions for the most advanced and gifted students, such as IMO, EGMO, and Romanian Masters. Besides, there are some unique Ukrainian competitions, such as: "Young Maths Tournament" (team research), "Champions Tournament" (including Maths, Physics, and Computer Science), and Kiev international Maths and Physics Fest (with the participation of scientists from the Maths Institute of the National Academy of Science in Ukraine).

2.4.5. Students' scientific research

It is very important to mention that the problems of these competitions are often the beginning of scientific research. For instance, the problem, given by prof. Valentine Leyfura, was the starting point for our cooperative research with my student Julia Fil. Consider the triangle ABC, with the points D, E, F belonging to the sides AB, BC, AC respectively. Investigate the perimeter of the given triangle ABC, DEF, using p, r, R — half of the perimeter, the inradius and the circumradius.

This project was presented at the National competition of students' scientific research. We found non-trivial lower and upper bounds for different cases, represented below.

Problem 2.2. *Consider the triangle ABC, , with the points D, E, F belonging to the sides AB, BC, AC respectively; p, S, r, R — half of the perimeter, the area, the inradius and the circumradius. Then:*

(1) $\frac{2pr}{R} \le DE + EF + DF \le p$, *given D, E, F — points of tangency of the circle, inscribed in the triangle ABC*;

(2) $3\sqrt{3}r \le DE + EF + DF \le \frac{3\sqrt{3}R}{2}$, *given* D, E, F *— foots of the angular bisectors of the triangle* ABC;

(3) $DE + EF + DF = \frac{2S}{R}$ *(for any right or acute triangles);*

$\frac{2S}{R} < ED + DF + EF \le \frac{3\sqrt{3}}{2}R$ *(For obtuse triangles),*

given D, E, F *– foots of the altitudes of the triangle* ABC;

(4) $\frac{2S}{R} \le DE + EF + DF < 3\sqrt{3}R$, *given* D, E, F *— points of tangency of the excircles, points* D, E, F *belong to the sides* AB, BC, AC *respectively.*

For example, let us find the upper bound in the inequality (4). First, find the sides and perimeter of the triangle DEF (Fig. 7)

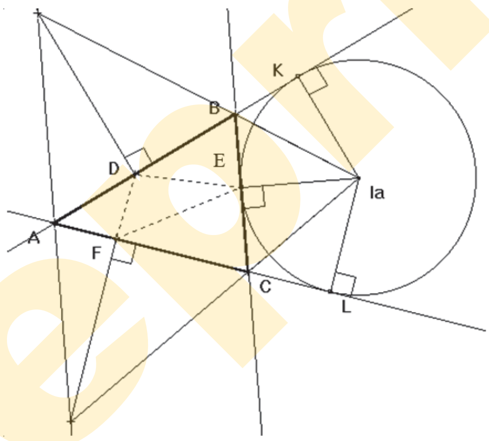

Fig. 7. The fourth case of the problem 2.2

BI_a is the angular bisector of $\angle KBC$ and CI_a is the angular bisector of $\angle LCB$, since I_a is the excenter relative to the vertex A.

So, $\angle ECI_a = \frac{1}{2}\angle ECL = 90° - \frac{\gamma}{2} \Rightarrow \angle EI_aC = \frac{\gamma}{2}$. In the same way, $\angle EI_aB = \frac{\beta}{2}$. Hence, from right triangles ΔBI_aE and ΔCI_aE, we have

$$EC = r_a \tan\frac{\gamma}{2}, \quad BE = r_a \tan\frac{\beta}{2}.$$

Notice that $r_a \tan\frac{\beta}{2}\tan\frac{\gamma}{2} = r$. Indeed, $BC = r_a\left(\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right)$. But, obviously,

$$BC = r\left(\cot\frac{\beta}{2} + \cot\frac{\gamma}{2}\right) = r\cdot\frac{\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}}{\tan\frac{\beta}{2}\tan\frac{\gamma}{2}},$$

so $r_a \tan\frac{\beta}{2}\tan\frac{\gamma}{2} = r$. Thus,

$$EC = \frac{r}{\tan\frac{\beta}{2}},\quad BE = \frac{r}{\tan\frac{\gamma}{2}}.$$

In the same way,

$$BD = \frac{r}{\tan\frac{\alpha}{2}},\quad AD = \frac{r}{\tan\frac{\mathrm{\beta}}{2}},\quad CF = \frac{r}{\tan\frac{\alpha}{2}},\ \text{and } AF = \frac{r}{\tan\frac{\gamma}{2}}.$$

According to the Cosine Theorem:

$$DE^2 = DB^2 + BE^2 - 2BD\cdot BE\cdot\cos\beta$$

$$= \frac{r^2}{\tan^2\frac{\alpha}{2}} + \frac{r^2}{\tan^2\frac{\gamma}{2}} - 2\cdot\frac{r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}}\cdot\cos\beta$$

$$= \frac{r^2}{\tan^2\frac{\alpha}{2}} + \frac{r^2}{\tan^2\frac{\gamma}{2}} + \frac{2r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}} - \frac{2r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}} - 2\cdot\frac{r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}}\cdot\cos\beta$$

$$= \left(\frac{r}{\tan\frac{\alpha}{2}} + \frac{r}{\tan\frac{\gamma}{2}}\right)^2 - 2\cdot\frac{r^2}{\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}}\cdot 2\cos^2\frac{\beta}{2}$$

$$= b^2 - \frac{4r^2\cos^2\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}} = b^2 - \frac{2r^2\sin\beta\cos\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}$$

$$= b^2 - \frac{2r^2\sin\beta\cdot\frac{p}{4R}}{\frac{r}{4R}} = b^2 - 2pr\sin\beta.$$

Here we used the formulas[4]

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{r}{4R} \quad\text{and}\quad \frac{p}{4R} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

[4] V. Prasolov (2001)

Note that if we multiply these two formulas, we obtain $\frac{1}{8}\sin\alpha\sin\beta\sin\gamma = \frac{pr}{16R^2}$. Then a useful corollary follows:

$$\sin\alpha\sin\beta\sin\gamma = \frac{S}{2R^2}.$$

Using the Sine Formula for the area of a triangle, we have

$$\begin{aligned} & b^2 - 2pr\sin\beta = b^2 - 2S\sin\beta = b^2 - ac\sin^2\beta \\ & = 4R^2\sin^2\beta - ac\sin^2\beta = 4R^2\sin^2\beta - 4R^2\sin\alpha\sin\gamma\sin^2\beta \\ & = 4R^2\sin^2\beta(1-\sin\alpha\sin\gamma). \end{aligned}$$

Therefore, $DE = 2R\sin\beta\sqrt{1-\sin\alpha\sin\gamma}$.

With the same way we get,

$$EF = 2R\sin\gamma\sqrt{1-\sin\alpha\sin\beta} \quad \text{and} \quad DF = 2R\sin\alpha\sqrt{1-\sin\beta\sin\gamma}.$$

To get the formula for the perimeter of the given triangle DEF we make an estimation using the Cauchy-Schwarz inequality:

$$\begin{aligned} & DE + EF + DF \\ & = 2R(\sin\beta\sqrt{1-\sin\alpha\sin\gamma} + \sin\gamma\sqrt{1-\sin\alpha\sin\beta}) + \sin\alpha\sqrt{1-\sin\beta\sin\gamma}) \\ & \le 2R\sqrt{\sin^2\alpha + \sin^2\beta + \sin^2\gamma}\cdot\sqrt{3-\left(\sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\alpha\sin\gamma\right)}. \end{aligned}$$

Notice, that from the famous Leibniz formula $MO^2 = R^2 - \frac{a^2+b^2+c^2}{9}$ we obtain the following Leibniz inequality: $a^2+b^2+c^2 \le 9R^2$, and therefore

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{a^2}{4R^2} + \frac{b^2}{4R^2} + \frac{c^2}{4R^2} \le \frac{9R^2}{4R^2} = \frac{9}{4}.$$

Now, using the AM-GM inequality we get:

$$\sin\alpha\sin\beta + \sin\beta\sin\gamma + \sin\alpha\sin\gamma \ge 3\cdot\sqrt[3]{\left(\sin\alpha\sin\beta\sin\gamma\right)^2} = 3\cdot\sqrt[3]{\left(\frac{S}{2R^2}\right)^2}.$$

Therefore, $DF + DE + EF \le 2R\cdot\frac{3}{2}\cdot\sqrt{3-\sqrt[3]{\left(\frac{S}{2R^2}\right)^2}} < 3\sqrt{3}R.$ □

Notice that the first expression for the upper bound is exact, but really long and complicated. The last expression $3\sqrt{3}R$ is a good estimation too — not really exact, but it is shorter.

2.4.6. *Maths Olympiads and scientific ideas*

It is necessary to say that Olympiads and other Maths competitions are important in some aspects. As it was already said, competitive and game forms of learning encourage students' motivation at lessons. Besides that, scientific tournaments are a perfect starting point for the first scientific research. But there are also deep problems that show the connection of solving Maths problems very quick and serious scientific ideas. It is obvious that the difficulties of these problems depend on the difficulty level of the Olympiad. But involving these scientific ideas is one of the most essential parts of Maths competitions. Let us have a look at applying these ideas of work with space basis and with functional equations. This unusual problem (author: Oleg F. Kryzhanovskiy, NYC) was given at Kharkiv Region Maths Olympiad[5].

Problem 2.3. *Let us name the sum of triangles with sides* $a_1 \le b_1 \le c_1$ *and* $a_2 \le b_2 \le c_2$ *the triangle with sides* $a_1 + a_2$, $b_1 + b_2$, and $c_1 + c_2$. *Name the product of the real number* $x > 0$ *and the triangle with sides* a, b, c *the triangle with sides* xa, xb, xc. *Find all functions with a set of triangles as the domain, and a set of real numbers as the range, with following properties:*

(1) for any triangles T_1, T_2: $f(T_1 + T_2) = f(T_1) + f(T_2)$ *("additive property");*

(2) for any triangle T and any real number $x > 0$: $f(xT) = xf(T)$ *("homogeneous property").*

Justify your answer.

Each triangle is defined by ordered triple of positive numbers (a,b,c), where $a \le b \le c$, which represent the triangle's sides. Try to guess the answer, using an analogy of vector components notation: $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$.

Thereby,

$$\begin{aligned} f(a,b,c) &= f(a(1,0,0) + b(0,1,0) + c(0,0,1)) \\ &= f(a(1,0,0)) + f(b(0,1,0)) + f(c(0,0,1)) \\ &= af(1,0,0) + bf(0,1,0) + cf(0,0,1) = xa + yb + zc. \end{aligned}$$

However, the "basis" consists of triangles "degenerated" into segments.

It would be easily improved by an operation, inverse to triangle addition — "subtraction of triangles". Indeed, if we have the equalities

$$(1,0,0) = (3,3,3) - (2,3,3), \quad (0,1,0) = (2,3,3) - (2,2,3),$$

and

$$(0,0,1) = (2,2,3) - (2,2,2),$$

then

$$(a,b,c) = a((3,3,3) - (2,3,3)) + b((2,3,3) - (2,2,3)) + c((2,2,3) - (2,2,2)).$$

Since "subtraction of triangles" is not defined, amend last equality by shifting "subtraction" with addition:

$$(a,b,c) + a(2,3,3) + b(2,2,3) + c(2,2,2) = a(3,3,3) + b(2,3,3) + c(2,2,3), \quad \text{or}$$
$$(a,b,c) + a(2,3,3) + b(2,2,3) + 2c(1,1,1) = 3a(1,1,1) + b(2,3,3) + c(2,2,3).$$

[5] O. Kryzhanovskiy (2012)

Use the function f to both parts of the last equality and apply its “homogeneous” and “additive” properties

$$f(a,b,c)+af(2,3,3)+bf(2,2,3)+2cf(1,1,1)=3af(1,1,1)+bf(2,3,3)+cf(2,2,3).$$

So we get

$$f(a,b,c)=a(3f(1,1,1)-f(2,3,3))+b(f(2,3,3)-f(2,2,3))+c(f(2,2,3)-2f(1,1,1)).$$

Thus, $f(a,b,c)=xa+yb+zc$, where x,y,z are any real numbers.

Checking by substitution of the given type function shows, that they satisfy the given.

Finally, we obtained the answer: $f(a,b,c)=xa+yb+zc$, where x,y,z are any real numbers.

3. Summary

So, the main idea of the given experience is a selection of gifted students, complex development of their mathematical abilities and encouraging students’ motivation to study Maths Science at school and university. The following results show the obvious success of the given method for 25 years in Academic Gymnasium No. 45, Kharkiv, Ukraine:

- More than 300 winners of Kharkiv Region Maths Olympiad;
- More than 50 winners of the final level of the National Ukrainian Math Olympiad;
- 3 winners of IMO (2003, Tokyo (Japan); 2011, Amsterdam (the Netherlands); 2018, Kluzh-Napoka (Romania)).

For instance, the silver medal winner of IMO - 2011 Olexii Kislinskij also won the International Mathematics Competition for University Students with Gold medal in 2012. Recently he graduated from Yale University (the USA) with a PhD in Maths in 2021.

- 1 winner (Gold Medal) of EGMO (2017, Zurich (Switzerland))
- More than 80% of school graduates enter Ukrainian and foreign universities on specialties connected with Maths and Computer Science.

The students of Kharkiv schools - the population of the city is 1 500 000 — have even more impressive results. For instance, every year some students from Kharkiv become winners of the IMO.

But the main result of my work is the creation of the gifted student’s mindset, involvement of them in the world of scientific research, Computer Science and IT, and forming them as integrated personalities.

Certainly, it is possible to use my experience at Academic Gymnasium No. 45, Kharkiv, Ukraine in other schools. What we need is a cooperation between students and their parents, teachers and school’s senior management, city authorities and different additional mathematical educational centers. Following this scheme of

Complex Maths education this experience might be useful for teachers of other schools. But it doesn't mean that this scheme should be followed absolutely accurate.

Of course, it should be adapted to the actual teachers' approaches. Thus, it helps them to succeed in their work and get great results from their students in Maths education.

References

A. Kryzhanovskiy (2015). Preparation for academic mathematics competitions as a factor of comprehensive teaching of mathematics in schools / Problems and prospects in professional education of the math teacher. *International Scientific and Practical Conference*. November 26–27, Vinnytsia, Ukraine, 2015 (pp. 181–183).

O. Kryzhanovskiy (2016). *Геометрія: підруч. Для 8 кл. загальноосвіт.* навчальних закладів [А. П. Єршова, В. В. Голобородько, О. Ф. Крижановський, С. В. Єршов]. — Харків: Вид-во Ранок, 2017. (A. Yershova, V. Goloborodko, O. Kryzhanovskiy, S. Yershov, *Geometry, Textbook for 8th Grade:* Ranok, 2016, Kharkiv.)

O. Kryzhanovskiy (2012). *Харьковские математические олимпиад В двух частях. Часть 1* / О. Ф. Крижановский, С. Ю. Игнатович, Т. М. Бойко. — Харьков: ХНУ имени В. Н. Каразина, 2012 (O. F. Kryzhanovskiy, S. Y. Ignatovich, T. M. Boyko. *Kharkov Mathematical Olympiads:* Kharkiv National Karazin University, 2012, Kharkiv.)

V. Prasolov (В. В. Прасолов) (2001). *Задачи по планиметрии.* — 4-е изд., дополненное — Москва: Издательство Московского Центра Непрерывного Математического Образования, 2001. (V. Prasolov. *Planimetry Problems*: MCCME, 2001, Moscow.)